Quantum Critical Behavior of Two Coupled Bose-Einstein Condensates

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Abstract: The quantum critical behavior of the Bose-Hubbard model for a description of two coupled Bose-Einstein condensates is studied within the framework of an algebraic theory. Energy levels, wave-function overlaps with those of the Rabi and Fock regimes, and the entanglement are calculated exactly as functions of the phase parameter and the number of bosons. The results show that the system goes though a phase transition and that the critical behavior is enhanced in the thermodynamic limit.

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As is well-known, the two-site Bose-Hubbard model[1] can be used to describe pair tunneling between two superconductors through an insulating junction, trapped ultra-cold bosonic gases, etc. Furthermore, it can also be used to prepare macroscopically entangled states.[2] The two-site Bose-Hubbard model has been investigated widely by many authors using various methods, such as the Gross-Pitaevskii approximation,[3] mean-field theory,[4−5] the quantum phase model,[6] and the Bethe ansatz method.[7] In [8], the temporal evolution of the expectation value for the relative number of particles between the two condensates for different choices of the coupling parameter and distinct initial states is analyzed. Also, quantum phase transitions that occur at zero temperature as a function of a coupling constant have become important in connection with various quantum many-body systems, such as the quantum Ising and rotor models,[9] Fermi liquids,[10], and atomic nuclei.[11] There are distinct features in these systems at the critical points. The main purpose of the present paper is to study the critical behavior of the two-site Bose-Hubbard model as a function of the coupling parameter and the total number of bosons.

Specifically, we consider the two-site Bose-Hubbard Hamiltonian with

$$\hat{\mathcal{H}} = -E_J(c^\dagger d + d^\dagger c) + E_c(c^\dagger c c^\dagger c + d^\dagger d d^\dagger d), \tag{1}$$

where $c^\dagger$ (c) and $d^\dagger$ (d) are boson creation (annihilation) operators in two traps or different hyperfine states. E_J is related to the Josephson coupling exchanging bosons between the two states, and E_c is related to the charging energy. In the present work we focus on the case where the effective interaction energy for the internal Josephson dynamics is negative,[12] $E_c < 0$. In order to study the transitional patterns of the system, the Hamiltonian (1) is re-parameterized as

$$\hat{H} = \hat{\mathcal{H}}/E_0 = -(1-x)(c^\dagger d + d^\dagger c) - \frac{4x}{n+1}(c^\dagger c c^\dagger c + d^\dagger d d^\dagger d), \tag{2}$$

where E_0 is a constant in arbitrary unit, n is the total number of bosons, and $0 \le x \le 1$ is the phase parameter. The division of the second term in (2) by $n+1$ serves to ensure that the boson rank of the two terms in $\hat{H}$ is the same, thereby making comparisons of results as a function of boson number more meaningful. For large values of n, this system is in the Rabi regime when $x \sim 0$, the Fock regime when $x \sim 1$, and the Josephson regime when $0 < x < 1$.

When $x = 0$, the system is in the Rabi regime with eigenstates given by

$$|x = 0; n_1, n_2\rangle = \frac{1}{2^{n/2}}(c^\dagger + d^\dagger)^{n_1}(c^\dagger - d^\dagger)^{n_2}|0\rangle, \tag{3}$$

where $n = n_1 + n_2$, and $|0\rangle$ is the boson vacuum state which is never degenerate. The corresponding eigenenergy of (2) is

$$E_{n_1,n_2}(x = 0) = n_2 - n_1. \tag{4}$$

When $x=1$, the system is in the Fock regime with eigenstates given by

$$|x=1;n_1,n_2\rangle = \frac{c^{\dagger n_1}d^{\dagger n_2}}{\sqrt{n_1!n_2!}}|0\rangle, \tag{5}$$

which is two-fold degenerate under the permutation $n_1 \rightleftharpoons n_2$. The corresponding eigenenergy is

$$E_{n_1,n_2}(x=1) = -\frac{4}{n+1}\left(n_1^2+n_2^2\right). \tag{6}$$

In the Josephson regime, $0<x<1$, the unitary transformation [13]

$$c = \frac{1}{\sqrt{2}}(a-ib),\ d = \frac{1}{\sqrt{2}}(a+ib) \tag{7}$$

can be used to rewrite Hamiltonian (2) as

$$\hat{H} = (1-x)(b^\dagger b - a^\dagger a) + \frac{2x}{n+1}S^+(0)S^-(0) - \frac{4n^2}{n+1}x, \tag{8}$$

where

$$S^+(0) = b^{\dagger 2} + a^{\dagger 2},\ S^-(0) = b^2 + a^2 \tag{9}$$

are boson pairing operators. In a manner similar to what was done in [7] and [14], it can be shown that the Bethe ansatz eigenvectors used for diagonalizing the Hamiltonian (8) may be written as

$$|x;\zeta;n=2k+\nu_1+\nu_2\rangle = \mathcal{N}S^+(y_1^{(\zeta)})S^+(y_2^{(\zeta)})\cdots S^+(y_k^{(\zeta)})|\nu_1,\nu_2\rangle, \tag{10}$$

where $\mathcal{N}$ is the normalization constant, $|\nu_1,\nu_2\rangle = a^{\dagger\nu_1}b^{\dagger\nu_2}|0\rangle$ with $\nu_i=0$ or 1 for $i=1,2$ is the boson pairing vacuum state that satisfies

$$a^2|\nu_1,\nu_2\rangle = b^2|\nu_1,\nu_2\rangle = 0, \tag{11}$$

and

$$S^+(y_i^{(\zeta)}) = \frac{b^{\dagger 2}}{1-y_i^{(\zeta)}} + \frac{a^{\dagger 2}}{1+y_i^{(\zeta)}}, \tag{12}$$

in which $y_i^{(\zeta)}$ $(i=1,2,\cdots,k)$ are spectral parameters that are to be determined, and ζ is an additional quantum number for distinguishing different eigenvectors with the same quantum number k. It can then be verified by using the corresponding eigen-equation that (10) is the solution when the spectral parameters $y_i^{(\zeta)}$ $(i=1,2,\cdots,k)$ satisfy the following set of equations:

$$\frac{2x}{n+1}\left(\frac{2\nu_2+1}{1-y_i^{(\zeta)}} + \frac{2\nu_1+1}{1+y_i^{(\zeta)}}\right) = \frac{1-x}{y_i^{(\zeta)}} + 8x\sum_{j(\neq i)}\frac{y_j^{(\zeta)}}{y_i^{(\zeta)}-y_j^{(\zeta)}} \tag{13}$$

for $i=1,2,\cdots,k$ with the corresponding eigen-energy given by

$$E_k^{(\zeta)}(x) = (1-x)\sum_{i=1}^{k}\frac{2}{y_i^{(\zeta)}} + (1-x)(\nu_2-\nu_1) - \frac{4n^2}{n+1}x. \tag{14}$$

It should be noted that the solutions (13) and (14) are not valid in the $x=0$ (Rabi) or 1 (Fock) limits.

To explore transitional patterns, some low-lying energy levels of the system for $x\in[0,1]$ with $n=10$, 40, 100, and 160, respectively, were calculated. The results, which are shown in Fig. 1, clearly show that there is a minimum in the excitation energies around $x\sim 0.25-0.35$ which corresponds to the Josephson critical region. While for small boson numbers the critical region is not very well defined, it becomes

clearer and sharper with increasing n values. Also, the energy level density in the critical region increases with increasing n.

To probe the nature of the critical point behavior more deeply, overlaps $|\langle x;n|x_0;n\rangle|$ for the ground states with $x_0 = 0$ or 1 for $n = 10$, 40, 80, and 120, respectively, were calculated. The results, which are given in Fig. 2, show that there is a crossover point at a certain nonzero amplitude for the overlaps $|\langle x;n|x_0 = 0;n\rangle|$ and $|\langle x;n|x_0 = 1;n\rangle|$ when n is relatively small, which yields to a cross-over region with near zero amplitude when n becomes large. This critical region in the large n limit should be regarded as a two phase (Rabi and Fock) coexistence region in analogue to the situation occurring in other finite boson systems.[11,15] Furthermore, there is a sharp change in $|\langle x;n|x_0 = 0;n\rangle|$ around a critical point $x_c \sim 0.25$ for large n. These results suggest that the largest absolute value of the derivative of $|\langle x;n|x_0 = 0;n\rangle|$ with respect to x occurs around the critical point in the thermodynamic limit.

As is known, the entanglement measure for any pure bipartite system is defined by

$$\eta = -\mathrm{Tr}(\rho_c \log_N \rho_c) = -\mathrm{Tr}(\rho_d \log_N \rho_d), \tag{15}$$

where ρ_c is the reduced density matrix obtained by taking the partial trace over the subsystem d, and similarly for ρ_d, and N is the total number of the Fock states for given n. We use the logarithm to the base N instead of base 2 to ensure that the maximal measure is normalized to 1. Fig. 3 shows the entanglement measure of the system as a function of x for $n = 10$, 40, 80, and 120, respectively. It is obvious that there is always a peak in the measure at or near the critical point, which is consistent with the so-called critical point entanglement.[16] For small n, the maximal value of the measure is near 1, while it decreases with increasing values of n. The peak grows sharper with increasing values of n, tracking the behavior of the overlap $|\langle x;n|x_0 = 0;n\rangle|$ and the excitation energies around the critical point.

In summary, we have studied the quantum critical behavior of the two-site Bose-Hubbard model within the framework of an algebraic theory. Energy levels, overlaps of the wavefunctions with those in both the Rabi and Fock regimes, and the entanglement were calculated exactly as a function of the phase parameter and the total number of bosons. The results show not only the quantum phase transition patterns of the model, but also that the critical behavior are greatly enhanced in the thermodynamic (large n) limit. This enhancement of critical phenomena with increasing of n should be common in other finite quantum many-body systems, as recently shown in the interacting boson model for atomic nuclei.[17] More importantly, the results suggest that while the quantum phase transition of the two-site Bose-Hubbard model between the Rabi and the Fock regimes is rather smooth for small values of n, it becomes clearer and sharp in the thermodynamic limit. Therefore, such quantum phase transition should be observed macroscopically.

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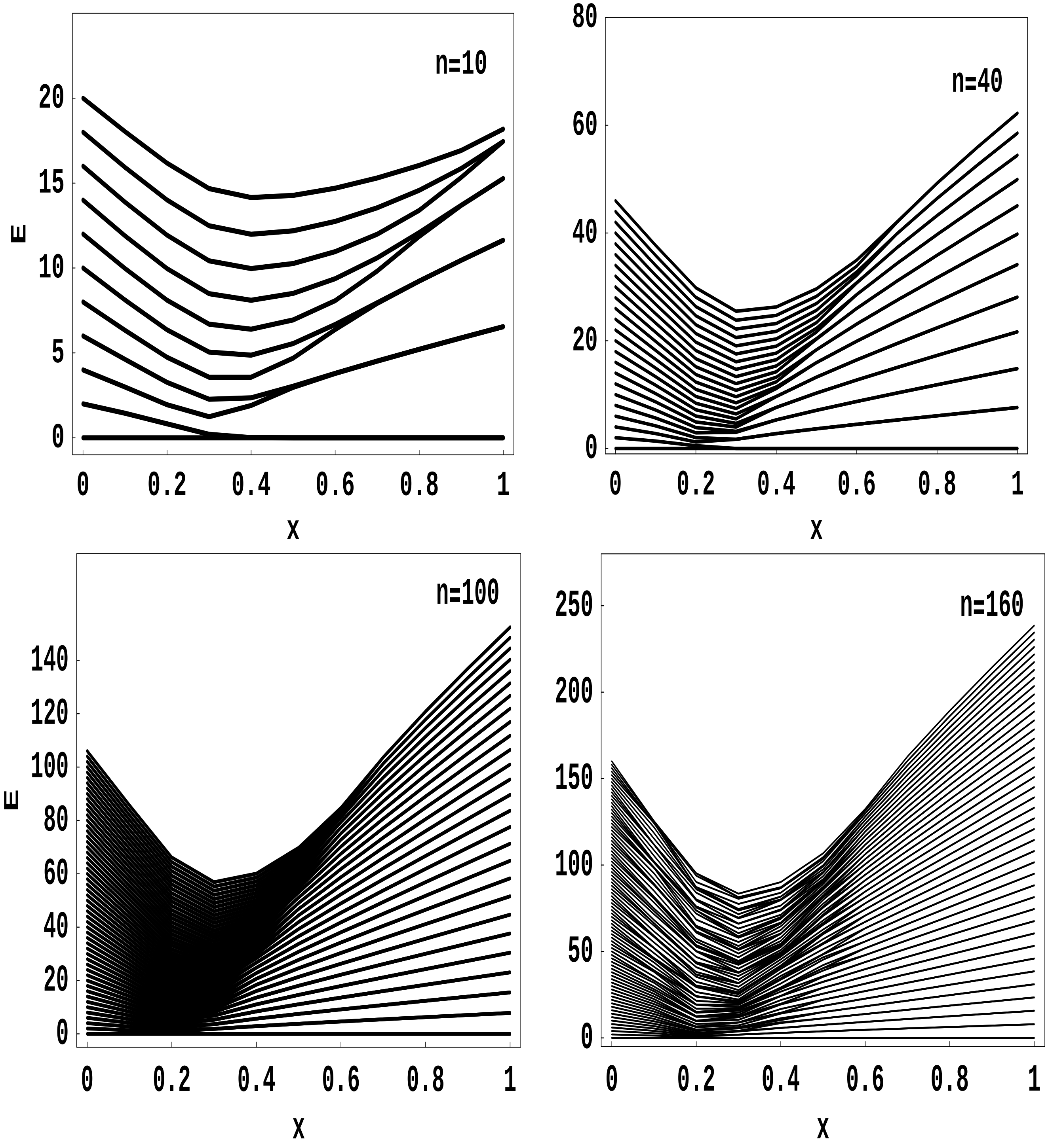

Fig. 1. Some low-lying energy levels of the two-site Bose-Hubbard model with the Hamiltonian given by (2) as functions of x for $n = 10$, 40, 100, and 160, respectively.

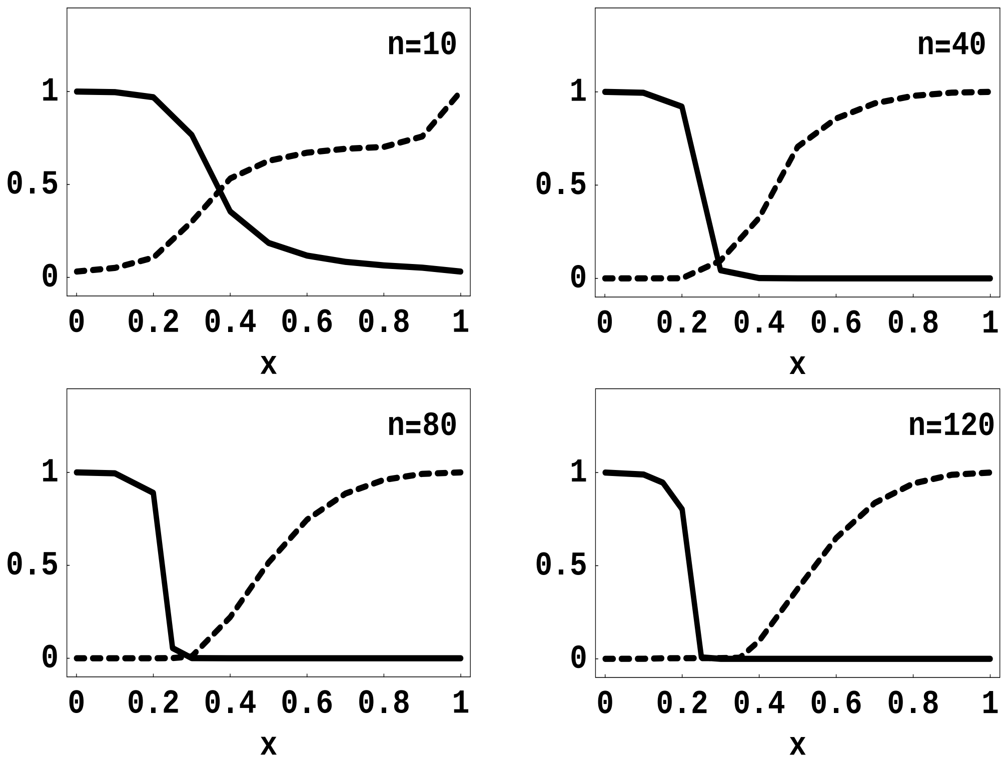

Fig. 2. Overlaps $|\langle x; n|x_0; n\rangle|$ for the ground states with $x_0 = 0$ or 1 for $n = 10$, 40, 80, and 120, respectively. The full lines represent the curves of $|\langle x; n|x_0 = 0; n\rangle|$, and the dotted lines represent those of $|\langle x; n|x_0 = 1; n\rangle|$.

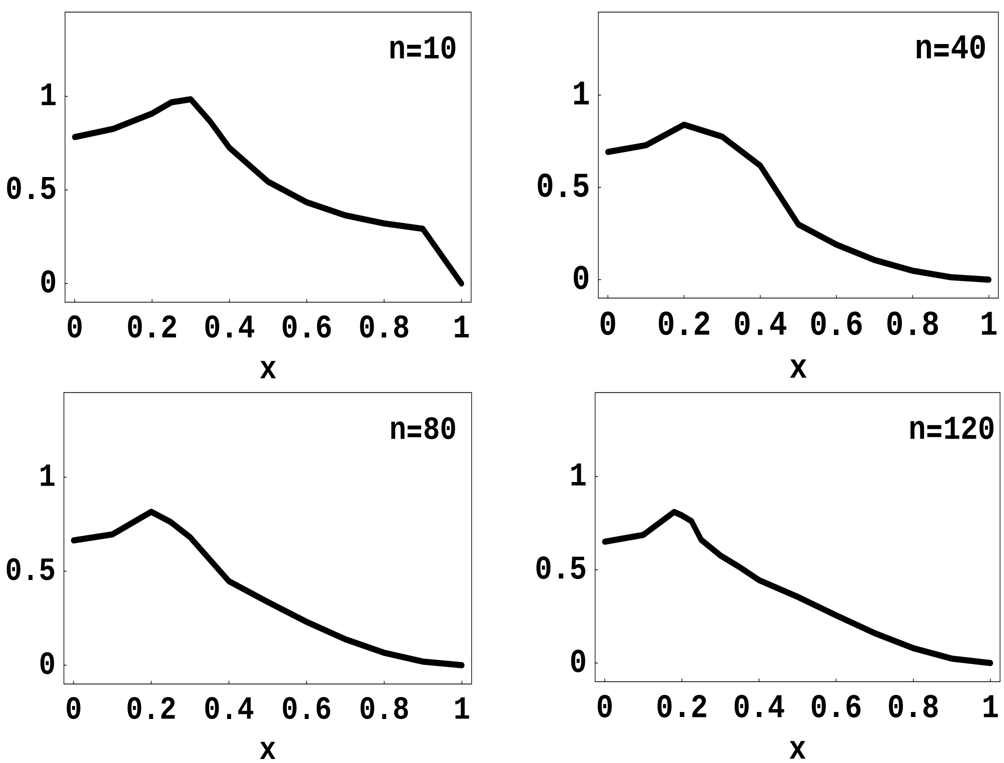

Fig. 3. The entanglement measures of the model as a function of x for $n = 10$, 40, 80, and 120, respectively.